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1 Introduction
Here is a list of my ideas about tropical geometry.

2 Tropical Hilbert polynomial
The most known example of non-realizable tropical curve is a tropical elliptic cubic that is not planar. One can compute tropical Hilbert polynomial, for realizable curves it coincides with the pre-limiting.

3 Tropical Weyl’s theorem
Theorem. Let $X$ be a complex curve and $f, g$ are two meromorphic function on it. Then $\prod_{x \in X} f(x)^{ord_x} = \prod_{x \in X} g(x)^{ord_x}$.

The same fact is true in the tropical setup. Proof. It is obvious for polynomials on tropical line $\mathbb{T}P^1$ (essentially as in the complex case). For general curve we do some local transformations of a curve and invariance of desired sum satisfied by mentioned fact for tropical line.

There is a proof by topical 1-forms and residues.

4 Tropically embeddable graphs
Having an abstract tropical curve $C = (V, E)$ ($V$ is a set of vertices, $E \subset V \times V, (v_1, v_2) \in E \iff (v_2, v_1) \in E, \forall v \in V, (v, v) \notin E$), graph has positive integer multiplicities on edges, $m: E \to \mathbb{N}$ its presenting as a tropical curve embedded in $\mathbb{T}^3$ consists of two stages:

Step 1: Choosing local behaviour in each vertex $v$, i.e. we have to assign to each edge in $v$ some integer non-zero vector $(emb: E \to \mathbb{Z}^3 \setminus \{0,0,0\}, emb((v_1, v_2)) = -emb((v_2, v_1)))$, and balancing condition should be satisfied ($\forall v \in V, if deg(v) > 1$ then $\sum m(e)emb(e) = 0$).

Step 2: Choosing length for each edge of $C (l: E \to \mathbb{R}_+, l(v_1, v_2) = l(v_2, v_1); \forall v \in V, if deg(v) = 1, (v, w) \in E$ then $l(v, w) = \infty$).

It is clear that sometimes after Step 1 we can’t perform Step 2.

Example 4.1. Let’s assume that graph $C$ contains a triangle $v_1v_2v_3$. If we’ve chosen integer vectors $emb(v_1, v_2), emb(v_2, v_3), emb(v_3, v_1)$ such way that they don’t lie in one plane that we can’t perform Step 2.

Definition 4.2. Graph $C$ after performing Step 1 we call stared graph, because we provide each vertex with star.

Definition 4.3. Stared graph $C$ is tropically embeddable if we can perform for him Step 2.

The question is does local embeddability imply global embeddability?

The answer is “no”.

Definition 4.4. $B_r(v) = \{ w \in V | \rho(w, v) < r \}$ with all edges between these vertices. For each $e_i = (v_i, w_i) \in E, v_i \in B_r(v), w_i \notin B_r(v)$ we add to $B_r(v)$ edge $(v_i, g_i)$, will be a new 1-valent vertex.

Definition 4.5. Stared graph $C$ in $N$-wild if $\forall v \in V, B_N(v) \subset C$ is tropically embeddable, but $C$ is not tropically embeddable.

Definition 4.6. Girth($C$) is the minimal length of cycle in $C$.

Definition 4.7. Cycle $A$ is reducible if there are two cycles $B, C$ such that $A \subset B \cup C$, $length(A) > max(length(B), length(C))$.

Definition 4.8. $\mu(C)$ is the maximal length of non-reducible cycle in $C$

Theorem 4.9. For any $N \in N$ there exists $N$-wild graphs.

Theorem 4.10. For any $N \in N, n \in N$ there exists $N$-wild graphs $C$, $girth(C) > n$.

Theorem 4.11. For any $n \in N$ there exists constant $m(n)$ that for any $N \in N$ there exists $N$-wild graphs $C$, $girth(C) > n, \mu(C) < m$.

Remark 4.12. It is clear that for stared graph $C$ we have linear conditions for its edges: $\forall (v_1, v_2, \ldots v_n, v_{n+1} = v_1), v_i \in V, (v_i, v_{i+1}) \in E, \sum_{e = (v_i, v_{i+1})} l(e)emb(e) = 0$. So, theorems imply that there are matrices which locally are solvable but globally are not consistent.

5 Generic tropical curves, singularities and resolutions
What does it mean when we speak about generic tropical curve in $\mathbb{T}^3$?

What is a singularity of tropical curve?

Definition 5.1. Resolution at vertex $v$ of tropical curve $C$ is a replacing small neighbourhood of $v$ by piece of tropical curve which has vertices with valency less that valency $v$.\(^1\)

\(^1\)One-valent vertices go to infinity as it is done in tropical geometry.
Definition 5.2. Vertex \( v \) is a singularity if one can do resolution at \( v \).

Theorem 5.3. Generic tropical curve (i.e. curve without singularities) in \( \mathbb{T}^3 \) have only 3-,4-,5-,6-,7-,8-,9-,10-valence vertices. Any vertex \( v, \deg(v) > 10 \) always can be resolved.

Idea of proof: Let’s consider singularities of tropical curves in \( \mathbb{T}^2 \). Any vertex \( v \) of tropical curve \( g \) in \( \mathbb{T}^2 \) corresponds to face \( F(v) \) in its Newton polygon \( \Delta_g \). Further, dimension of vertex resolution space equals number of integer point in the interior of \( F(v) \). Let’s consider vertex \( v \) of tropical curve \( C \) in \( \mathbb{T}^3 \). Project neighbourhood of \( v \) along on of its edges. Now we have picture in \( \mathbb{T}^2 \). We can resolve it and is rises question can we lift this resolution to \( \mathbb{T}^3 \)?

Lemma 5.4. Suppose that dimension of vertex resolution space equals 1. Then we can’t lift resolution to \( \mathbb{T}^3 \).

Proof: none of lifting of curves in one-parametric resolution family contains image of edge collapsed by projection.

Lemma 5.5. Any convex integer \( n-gon \) on the plane contains in interior at least 2 integer points if \( n > 6 \). There are examples of 3-,4-,5-,6-gons which contain only 1 integer point.

Denote by \( g(n) \) the minimal number of integer points in interior of \( n-gon \) with integer vertexes.

Lemma 5.6. \( g(n) < n \) for \( n < 10 \), \( g(10) = 10 \) and \( g(n) > n \) for \( n > 10 \) [See article On the first unknown value of the function \( g(v) \). Daria Olszewska]

Proof of the theorem. Let’s consider a vertex \( v, \deg(v) > 10 \) of a tropical curve in \( \mathbb{T}^3 \). The Newton polygon \( \Delta_v \) of projection of \( v \) is a \( n = \deg(v - 1) \)-gon, therefore \( \Delta_v \) has \( i \geq n \) vertexes in interior. We are looking for a projection resolution which can be lifted to \( \mathbb{T}^3 \). A lifting is just a assigning height to each vertex. Vertexes of \( \Delta_v \) are given by height automatically. In each of vertex we have the balancing condition that should be checked only for height coordinate. Therefore we have \( n+i \) conditions. But we have a \( i \) degrees of freedom for plane resolution and \( i \) degrees of freedom for height on interior points. Further, \( 2i \geq n+i \), therefore we always can resolve system of these equations.

Remark 5.7. One can have problems only with vertexes of \( \Delta_v \) because its balancing conditions can include none of interior vertexes. But we can decompose \( \Delta_v \) such way that each of two neighbours would be connected with two interior vertexes. It solves the problem.

Remark 5.8. Tropical geometry have sense only since there are connections with other branches of mathematics. So, tropical singularities need to be interpreted in realm of geometry (complex, lagrangian etc.) or algebra (Puiseux series, Berkovich spaces etc)

6 Tropical curve as intersection of tropical hypersurfaces

In the classical situation we know that any algebraic curve in \( \mathbb{R}^3 \) coincides with common zeroes of three polynomial.

Is it true in tropical world?

Observation 6.1. There is a neighbourhood of vertex degree 7 which can’t be presented as intersection of two tropical hypersurfaces

Idea of proof: let’s consider a small sphere around the vertex and intersect is with curve and hypersurfaces. On this sphere we have a collection of seven points and want to present their as intersection of two tropical curves on sphere. By Bezout Theorem we are seeking for line and curve of degree 7. So, just choose points which doesn’t contain in tropical line.

Theorem 6.2. There is a tropical curve \( C \) such that for any 3 tropical hypersurfaces \( F_1, F_2, F_3, C \subset F_1 \cap F_2 \cap F_3 \) it turns out that \( F_1 \cap F_2 \cap F_3 \setminus C \neq \emptyset \)

Idea of proof: Let’s consider the Borromean rings \( C_1 \cup C_2 \cup C_3 \) and pull on tropical curve. It is know that for any Seifert surfaces \( G_i, \partial G_i = C_i, G_1 \cap G_2 \cap G_3 \neq \emptyset \).

Lemma 6.3. Any tropical curve \( C \) contains weight 1 in a tropical surface that has only weight 1 faces

Proof: Let’s consider the projection of \( C \) onto x-axe. Unfortunately we can’t take product of projection and line as a desired surface, for example, edge \((1,2,2)\) projects to \((2,2)\) which is not primitive. But we can create new directions. So, that introduce additional intersection bat we can resolve them.