Galerkin method

Theory of partial differential equations (PDEs) has its origins in the 19th century when mathematicians began to study the problem: find a function \( u(x, y) \) such that

\[
\Delta u + f = 0 \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial \Omega, \quad (1)
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with smooth (or piecewise smooth) boundary \( \partial \Omega \). It was unknown how to solve this equation and how the solution could depend on the problem data. Moreover, the existence of a solution to this problem has also been the subject of active discussion.

Similar questions arose in connection with other problems \( Lu = f \) associated with a partial differential operator \( L \) mapping a Banach space \( X \) to a Banach space \( Y \). At that time, the equations were studied in the framework of classical analysis and, therefore, the space \( X \) was considered as the space of continuous functions having sufficient amount of classical derivatives. It was discovered that some problems have equivalent variational formulations, e.g., (1) is equivalent to minimization of the energy functional

\[
\int_{\Omega} \frac{1}{2} \left| \nabla u \right|^2 - fu \, dx.
\]

In 1909, W. Ritz suggested to find approximate solutions of variational problems in the form

\[
u_N(x) = \sum_{i=1}^{N} \alpha_i w_i(x), \quad (2)
\]

where \( N \) is a natural number and the coordinate functions \( w_i \) belong to \( X \) and form a linearly independent system. The weights \( \alpha_i \) should be chosen to minimize the functional.

However, many problems are not generated by a certain (energy) functional. It was necessary to create a unified and mathematically justified approach valid for differential equations of all types. Intuitively, it was clear that a suitable approximation \( v \) should somehow minimize the residual \( Lv - f \), but which form of the residual should be used and how to select the set of coordinate functions? Without right answers to these questions, it is impossible to prove that \( u_N \) converges to the exact solution \( u \).

The idea of Galerkin method is to find \( \alpha_i \) from the condition: residual must be orthogonal to a finite dimensional subspace of test functions with integral type orthogonality conditions. For the problem (1), this principle yields (after integration by parts)

\[
\int_{\Omega} \nabla u_N \cdot \nabla w_i \, dx = \int_{\Omega} fw_i \, dx \quad \forall w_i \in X_N, \quad (3)
\]

where \( X_N \) contains functions of the form (2) vanishing on the boundary \( \partial \Omega \). In more general cases, the sets of coordinate and test functions may not coincide and the orthogonality relation has the form

\[
\langle Lu_N - f, \eta \rangle = 0, \quad \forall \eta \in Y_M \quad (4)
\]

where \( u_N \) is defined by (1) and \( Y_M \) (dim \( Y_M = M \)) is a set of test functions, \( Y_M \subset Y' \), where \( Y' \) is the space conjugate to \( Y \) and \( \langle \cdot , \cdot \rangle \) denotes the duality pairing of \( Y \) and \( Y' \). Certainly, \( Y_M \) and \( X_N \) must be selected such that the system (4) is solvable.

This approach is very flexible. If \( X \) and \( Y \) are Hilbert spaces and the spaces \( X_N \) and \( Y_M \) coincide, then the method is called Bubnov–Galerkin. This name was used by S. Mikhlin who was the first to prove its convergence. If the integral orthogonality relations originates from the Euler equation generated by a quadratic functional, then we have at the Ritz–Galerkin method. More general schemes (as (4) were studied by G. Petrov who also extended the method to eigenvalue problems. Portraits of W. Ritz, B. Galerkin, I. Bubnov, and G. Petrov are presented below.
"How to guarantee existence of solutions to boundary value problems for PDEs?"

At the end of the 19th century this question was open. In particular, solvability of the simplest problem (1) was intensively discussed in the literature (typically in the classical sense, i.e., \( u \) was seeking in \( C^2(\Omega) \cap C(\Omega) \)). This question was answered after many years of studies that have completely reconstructed the theory of PDEs.

A new conception of generalized or weak solutions was created by D. Hilbert, H. Poincaré, S. Sobolev, R. Courant, O. A. Ladyzhenskaya, and many others outstanding mathematicians. In fact, the Galerkin method served as a turning point in the study of this problem. This is easy to observe with this paradigm the integral relation of (2).

Indeed, let us tend \( N \) to \( +\infty \). If we await that \( u_N \) will tend (in some sense) to the exact solution, then it is natural to consider the limit form of (2), where \( X_N \) is replaced by an infinite dimensional functional space \( \tilde{X} \) (which should be a proper closure of \( \{X_N\} \)). This way leads to the generalised statement of (1): find \( u \in \tilde{X} \) satisfying the boundary conditions such that

\[
\int_{\Omega} \nabla u \cdot \nabla w \, dx = \int_{\Omega} f w \, dx \quad \forall w \in \tilde{X}, \quad (5)
\]

Now we know that in the case of Lipschitz \( \Omega \) the closure generates the Sobolev space \( \tilde{H}^1(\Omega) \) (of functions vanishing on the boundary and having square summable generalized derivatives of the first order).

Integral type definitions of solutions to PDEs is nowadays commonly accepted (a systematic exposition can be found in O. A. Ladyzhenskaya and N. N. Ural'tseva, “Linear and Quasilinear Elliptic equations”).

If \( \tilde{X} \) is a reflexive Banach space, then from (3) it follows that \( \|\nabla u_N\|_0 \leq C\|f\|_0 \) with a constant independent of \( N \). Hence, there exists a subsequence of \( u_N \) weakly converging to a function \( u \in \tilde{X} \). Using this fact, it is not difficult to show that \( u_N \) converges to a function \( u \) satisfying (5). Similar arguments can be used in other boundary value problems. Therefore, Galerkin approximations suggest a method for proving existence of weak solutions.

E. Hopf used this idea in order to prove existence of weak solutions to nonstationary Navier–Stokes equations. A similar approach was often used by O. Ladyzhenskaya and N. N. Ural’tseva, for various nonlinear problems in the theory of viscous fluids and other PDEs.

It is worth noting that the concept of a generalised (weak) solution and the Galerkin method are closely related to the Virtual Work Principle in mechanics that dates back to J. D’Alembert (who used it for a mechanical system of rigid bodies) and J.-L. Lagrange (who suggested a generalization for continuum media problems). This principle was known already at the beginning of the 19th century. The development of mathematics at that time was insufficient to correctly determine what should be considered as “the set of virtual displacements” and properly state the corresponding boundary value problems.

In 1943, R. Courant suggested a version of the method with locally supported test functions, which generated geometrically flexible numerical schemes with dispersed resolving matrices. Later it was named the Finite Element Method (FEM). Mathematical analysis of FEM is based upon two fundamental relations: Galerkin orthogonality and projection estimate. The latter estimate forms the basis of error analysis. It states that the distance between \( u \) and \( u_N \) is controlled by the distance between \( u \) and the respective finite dimensional space \( X_N \).

In this or other form, the majority of modern computational technologies use approximations of the Galerkin type. For example, the Discontinuous Galerkin method uses (3) or (4) with discontinuous test functions. Many other methods (spectral, finite volume, weak Galerkin, isogeometric, meshless) can be viewed as advanced versions of the Galerkin concept. The need to calculate Galerkin approximations for real life scientific and engineering problems stimulated studies in numerical linear algebra and generated the creation of multigrid iteration methods and domain decomposition method (DDM).

Multigrid methods allow to solve very large systems of linear equations by using several different scales of discretizations in order to optimise the process of computations. Domain decomposition method originates from the method of Schwarz, who suggested to decompose domains with complicated geometry into a collection of simple subdomains (e.g., rectangles, convex polygons), for which the corresponding sets of test functions can be constructed by simple methods.
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In [7], the reader can find more about further development of the method and useful references.
A systematic consideration of the Galerkin method in application to elliptic, parabolic, and hyperbolic equations is presented in [8].
For Discontinuous Galerkin method (DG) see [1] and references therein.

References